STABILIZATION OF AN ALTERNATING-CURRENT-CARRYING PLASMA FILAMENT BY A QUADRUPOLE MAGNETIC FIELD

M. G. Nikulin

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Dynamic stabilization of straight and toroidal current-carrying plasma jets by a high-frequency quadrupole magnetic field was proposed by Osovets [1]. A more rigorous theoretical analysis of the problem was performed by Levin and Rabinovich [2], who obtained a Routh function for studying the dynamics of the filament in various magnetic fields for a thin filament experiencing longwave serpentine- and construction-type disturbances. In this paper, the method proposed in [2] is applied to the stabilization of a plasma filament in which (as distinct from [1, 2]) flows an alternating current while the quadrupole field is either constant, or varies at a high frequency, as in [1, 2].

1. The motion of thin, ideally conducting annulus along whose surface flow a transverse current I_1 and a longitudinal current I_2 can be described [2] by a Routh function,

$$R = \mathcal{L} - \frac{\Phi_1^2}{2L_1} - \frac{(\Phi^e - \Phi_2)^2}{2L_2} \qquad (\mathcal{L} = T - U). \tag{1.1}$$

where $\mathcal{L}$ is the mechanical Lagrangian of the annulus, T and U are the kinetic and intrinsic energies, Φ_1 is the field flux frozen within the plasma, Φ^e is the external-field flux through the annulus, Φ_2 is the total field flux through the annulus, and L_1 and L_2 are the inductances corresponding to the currents I_1 and I_2. Owing to ideal conductivity, the field fluxes Φ_1 and Φ_2 are conserved, i.e.,

$$\Phi_1 = c^{-1}L_1I_1 = \text{const}, \quad \Phi_2 = c^{-1}L_2I_2 + \Phi^e = \text{const}. \tag{1.2}$$

The Routh function (1.1) can be expressed solely in terms of mechanical variables, with respect to which it plays the role of an ordinary Lagrange function. For a known function (1.1), the derivation of the equations of motion and the stability analysis can be carried out by standard procedures.

Convenient for use are mechanical variables in the form $\varepsilon(\varphi)$, $\delta(\varphi)$, and $\alpha(\varphi)$, which describe the perturbation of the annulus

$$r(\varphi) = b\,[1 + \varepsilon(\varphi)], \quad z(\varphi) = b\delta(\varphi), \quad a_\sim = a\,[1 + \alpha(\varphi)].$$

where r, φ, and z are the cylindrical coordinates of the axial curve of the filament; a and b are the inner and outer radii of an equilibrium annulus, respectively; $a_\sim$ is the variable cross-sectional radius; ε and δ are quantities characterizing the horizontal and vertical deflections; and α is a quantity characterizing the constrictions of the filament.

By expanding the function ε, δ, and α into a Fourier series of the type

$$\varepsilon = \varepsilon_0 + \sum_{n=1} (\varepsilon_{nc} \cos n\varphi + \varepsilon_{ns} \sin n\varphi), \tag{1.3}$$

the functions T, U, $\Phi_1^2/2L_1$ and L_2 take the form [2]

$$T = \frac{1}{2} Mb^2 \Big\{ 2(\dot\varepsilon_0^2 + \dot\delta_0^2) + \frac{a^2}{b^2}\dot\alpha_0^2 + \sum_{n=1} \Big[\Big(1 + \frac{1}{n^2}\Big)\dot\varepsilon_n^2 + \dot\delta_n^2 + \\ + \frac{4}{n^2}(\dot\alpha_n^2 + \dot\varepsilon_n\dot\alpha_n) \Big] \Big\}, \tag{1.4}$$

$$U = U_0 - p_0V_0 \{\varepsilon_0 + 2\alpha_0 + 2\langle\varepsilon\alpha\rangle + \langle\alpha^2\rangle - {}^1/_2\gamma(\varepsilon_0 + 2\alpha_0)^2 + {}^1/_2\langle\varepsilon'^2 + \delta'^2\rangle\}, \tag{1.5}$$

$$\Phi_1^2 / 2L_1 = p_1V_0 \{1 + \varepsilon_0 - 2\alpha_0 - 2\langle\varepsilon\alpha\rangle + 3\langle\alpha^2\rangle + {}^1/_2\langle\varepsilon'^2 + \delta'^2\rangle\}, \tag{1.6}$$

$$L_2 = 2\pi b \{l + (l+2)\varepsilon_0 + \varepsilon_0{}^2 - 2\alpha_0 - 2\langle\varepsilon\alpha\rangle + \langle\alpha^2\rangle + + {}^1/_2 \sum_{n=1} n^2 [(\Lambda - g_r(n))\varepsilon_n{}^2 + (\Lambda - g_z(n))\delta_n{}^2]\}, \quad (1.7)$$

where

$$\varepsilon_n{}^2 = \varepsilon_{nc}{}^2 + \varepsilon_{ns}{}^2, \qquad \dot\varepsilon_n\dot\alpha_n = \dot\varepsilon_{nc}\dot\alpha_{nc} + \dot\varepsilon_{ns}\dot\alpha_{ns},$$

while δ_n^2, α_n^2, $\dot\varepsilon_n^2$, $\dot\delta_n^2$, and $\dot\alpha_n^2$ are of a similar structure; M and V_0 are the mass and equilibrium volume of the annulus; p_0 is the equilibrium gas pressure; and γ is the ratio of specific heats.

$$p_1 = B_1{}^2/8\pi, \quad B_1 = \Phi_1/\pi a^2, \quad \Lambda = \ln\frac{8b}{a}, \quad l = 2(\Lambda - 2),$$

$$g_r(n) = 2\left(1 - \frac{1}{4n^2}\right)\sum_{i=1}^{n}\frac{1}{2i-1} + \frac{1}{2} - \frac{1}{n^2},$$

$$g_z(n) = 2\left(1 - \frac{3}{4n^2}\right)\sum_{i=1}^{n}\frac{1}{2i-1} + \frac{1}{2}.$$

The brackets denote averaging over the azimuth angle φ,

$$\langle f(\varphi)\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(\varphi)\,d\varphi.$$

The functions T and U are obtained under the assumption of total adiabaticity, where the gas pressure is related with the total volume of the annulus and by the adiabatic equation. Because of the approach used in the derivation, formulas (1.4)–(1.7) have a physical meaning only for ε, δ, and α disturbances that are small compared to unity but are large compared to the a/b ratio. Moreover, the disturbances must be sufficiently smooth: the derivatives ε', δ', and α' with respect to φ may not exceed the order of magnitude of the functions themselves. This means that in expansions of the (1.3) type, the higher-order harmonics have an infinitesimal weight. Consequently, the deflection harmonics under consideration satisfy conditions of the form

$$na/b \ll n\Delta_n/b \ll 1. \quad (1.8)$$

where $\Delta_n \sim b\varepsilon_n \sim b\delta_n$ is the displacement of the filament.

Let us now determine the quantity $\Phi^e - \Phi_2$, which depends on the structure of the magnetic field. The external magnetic field in the proximity of the annulus represents a superposition of the main field, required to keep the annulus in equilibrium, and an additional quadrupole field, required for its stabilization. If the lines of force of the external field are symmetrical with respect to the plane z = o, and the main field has a value of $B_{0z} = B_0$ at the circumference r = b, z = 0, then in the region filled out by the annulus, the magnetic field can be represented with an accuracy to (r – b) terms quadratic with respect to z in the form

$$B_r = Gz, \quad B_\varphi = 0, \quad B_z = B_0 + G(r - b), \quad (1.9)$$

where

$$G = G_0 + G_q, \quad G_0 = \left(\frac{\partial B_{0z}}{\partial r}\right)_{\substack{r=b\\ z=0}}, \quad G_q = \left(\frac{\partial B_{qz}}{\partial r}\right)_{\substack{r=b\\ z=0}}, \quad (1.10)$$

while the subscripts 0 and q refer to quantities associated with the main and the quadrupole fields, respectively.

Field (1.9) is described by a vector potential with one component $A_\varphi = A$,

$$B_r = -\frac{\partial A}{\partial z}, \qquad B_z = \frac{\partial A}{\partial r} + \frac{A}{r} \quad (1.11)$$

By expanding A into a Taylor series in the proximity of an equilibrium filament, and using formulas (1.9)–(1.11) in combination with the equality $dl_\varphi = b(1 + \varepsilon)d\varphi$ in [2], for the external field flux through a perturbed annulus we obtain

$$\Phi^e = \int \mathbf{A}d\mathbf{l} = \Phi_0 + \Phi_q + \pi b^2 [2B_0\varepsilon_0 + B_0\langle\varepsilon^2\rangle + bG\langle\varepsilon^2 - \delta^2\rangle]$$

where Φ_0 and Φ_q are the fluxes of the main and quadrupole fields through the perturbed annulus. We examine the case in which the longitudinal current in the filament is

$$I_2 = I_{2c} \cos\omega t \tag{1.12}$$

and the quadrupole field is constant, assuming that

$$\Phi_0 = \Phi_{00} + \Phi_{0c} \cos\omega t, \qquad \Phi_q = \Phi_{q0} = \text{const}. \tag{1.13}$$

Then, the second relation in (1.2) yields

$$\Phi_{00} + \Phi_{0c} \cos\omega t + \Phi_{q0} - \Phi_2 = -2\pi blc^{-1} I_{2c} \cos \omega t. \tag{1.14}$$

Hence

$$\Phi_{00} + \Phi_{q0} - \Phi_2 = 0, \quad \Phi_{0c} = -2\pi blc^{-1} I_{2c}. \tag{1.15}$$

The first equation in (1.15) shows that the flux Φ_{00} is necessary to compensate for Φ_{q0} in the case in which the plasma annulus forms prior to application of the field, and consequently $\Phi_2 = 0$.

The field responsible for the flux Φ_{00} should be selected in such a way that it vanish in the region filled out by the perturbed annulus, for which Eqs. (1.9) and (1.10) hold. Then, obviously, $B_0 = B_{0c} \cos \omega t$, $G_0 = G_{0c} \cos \omega t$, i.e., B_0 and G_0 do not contain constant components. As distinct from G_0, in view of the constant quadrupole field, $G_q = G_{q0} =$ = const.

For further analysis it is convenient to introduce a mean field $B' = \Phi_{0c}/\pi b^2$. From the second equation in (1.15) it follows that the magnetic-field amplitude of current I_2,

$$B_2 = 2I_2 / ca = B_{2c} \cos \omega t$$

is related to B' by the simple expression

$$B_{2c} = -B'b / al. \tag{1.16}$$

After some simple but cumbersome transformations, we obtain

$$(\Phi^e - \Phi_2)^2 / 2L_2 = p_{2c} V_0 [(l + w)(1 + \cos 2\omega t) + 4(b/a)^2 \varkappa_{q0} \times \times \langle \delta^2 - \varepsilon^2 \rangle \cos\omega t], \tag{1.17}$$

where

$$\begin{gathered} w = (K-2)\varepsilon_0 + 2\alpha_0 + 2\langle \varepsilon\alpha \rangle - 2(l+4-K) l^{-1} \varepsilon_0 \alpha_0 + \\ + 4l^{-1}\alpha_0{}^2 + [{}^1/_4 (l+4-K)^2 l^{-1} - 1] \varepsilon_0{}^2 - \langle \alpha^2 \rangle + {}^1/_2 (l+K) \times \\ \times \langle \varepsilon^2 \rangle + 2(b/a)^2 \varkappa_0 \langle \delta^2 - \varepsilon^2 \rangle - {}^1/_2 \sum_{n=1} n^2 [(\Lambda - g_r(n)) \varepsilon_n{}^2 + \\ + (\Lambda - g_z(n)) \delta_n{}^2], \\ K = (4\chi - 1) l, \quad \chi = B_{0c} / B', \quad p_{2c} = B_{2c}{}^2 / 16\pi, \\ \varkappa_0 = aG_{0c}/B_{2c}, \quad \varkappa_{q0} = aG_{q0} / B_{2c}. \end{gathered} \tag{1.18}$$

By adding (1.5), (1.6), and (1.17) and equating to zero the time-independent coefficients of α_0 and ε_0, we arrive at conditions for which forced oscillations of the inner and outer radii of the annulus occur about the values of a and b,

$$p_0 = {}^1/_2 (K-1) p_{2c}, \quad p_1 = {}^1/_2 (3-K) p_{2c}. \tag{1.19}$$

Consequently, the parameter K must satisfy the inequalities

$$1 < K \leqslant 3. \tag{1.20}$$

Taking (1.19) into consideration, for the generalized potential energy of the annulus

$$W = U + \Phi_1{}^2 / 2L_1 + (\Phi^e - \Phi_2)^2 / 2L_2. \tag{1.21}$$

we obtain the expression

$$
\begin{aligned}
W = W_0 &+ p_{2c} V_0 \{[2(2-K) + \gamma(K-1) + 4l^{-1}]\alpha_0^2 + \\
&+ {}^1/_4 [(l+4-K)^2 l^{-1} - 4 + \gamma(K-1)]\varepsilon_0^2 + [\gamma(K-1) - \\
&- 2(l+4-K) l^{-1}]\varepsilon_0\alpha_0 + (2-K)\sum_{n=1}\alpha_n^2 + {}^1/_2 (l+K)\langle\varepsilon^2\rangle - \\
&- {}^1/_4 \sum_{n=1} n^2 [(l+K-q_r(n))\varepsilon_n^2 + (l+K-q_z(n))\delta_n^2] + 2(b/a)^2 \times \\
&\times (\varkappa_0 + 2\varkappa_{q0}\cos\omega t)\langle\delta^2 - \varepsilon^2\rangle + w\cos 2\omega t\},
\end{aligned}
\tag{1.21}
$$

where W_0 is the potential energy of the unperturbed annulus,

$$q_r(n) = 2[g_r(n) - 1], \quad q_z(n) = 2[g_z(n) - 1]$$

Formulas (1.4) and (1.21) yield the required Routh function R = T − W, which in principle permits complete analysis of the annulus motion. The problem is complicated, however, by the fact that the individual perturbation harmonics do not constitute normal oscillations of the system. It can be seen from (1.4), (1.18), and (1.21) that there exists a relationship between the oscillations about the outer and inner radii ($\varepsilon\alpha$-relationship). At the same time, the absence of $\delta\alpha$-relationship indicates that in the case of a straight filament, die harmonics are not related, i.e., that they are normal oscillations of the system. Indeed, by setting

$$x = b\varepsilon, \quad z = b\delta, \ \rho = a\alpha$$

and passing to the limit ($b \to \infty$ and $n \to \infty$) at a finite ratio n/b = k (which constitutes the perturbation wave number), for the instantaneous kinetic energy T_1 and instantaneous potential energy W_1 we obtain expressions in which all the harmonics are separated

$$T_1 = \frac{1}{2}\pi a^2\sigma\left\{\dot{x}_0^2 + \dot{z}_0^2 + \frac{1}{2}\dot{\rho}_0^2 + \sum_k\left[\frac{1}{2}(\dot{x}_k^2 + \dot{z}_k^2) + 2\dot{\rho}_k^2 (ka)^{-2}\right]\right\}, \tag{1.22}$$

$$
\begin{aligned}
W_1 = W_{01} &+ \pi p_{2c}\{[2(2-K) + \gamma(K-1)]\rho_0^2 + (2-K)\sum_k \rho_k^2 - \\
&- {}^1/_4 \sum_k (ka)^2 [2\ln(2/ka) - 2C + K - 3](x_k^2 + z_k^2) + 4\varkappa_{q0}\langle z^2 - x^2\rangle \times \\
&\times \cos 2\omega t + [2a\rho_0 - \langle\rho^2\rangle - {}^1/_4 \sum_k (ka)^2 [2\ln(2/ka) - 2C - 1](x_k^2 + \\
&+ z_k^2)]\cos 2\omega t\}
\end{aligned}
\tag{1.23}
$$

(C = 0.577... is the Euler constant).

Here, σ is the plasma density, while summation is performed over wave numbers k that satisfy, in accordance with (1.8), the conditions

$$ka \ll k\Delta_k \ll 1 .$$

Expression (1.23) lacks terms with B_0 and C_0, since for a straight filament, no external field beside the quadrupole field is required.

Let us now analyze the stability of a straight filament, keeping in mind the following circumstances: first, that for a straight filament, the individual perturbation harmonics are normal oscillations, so that the stability conditions obtained are conditions sufficient for stabilization, second, that in the analysis of a straight filament, all the intrinsic characteristics of a system with an alternating longitudinal current are retained (with the exception of the forced oscillations of the outer radius), and third, that toroidal geometry, as shown in [2], has an appreciable effect only on the stability conditions of harmonics with $n \sim 1$, where dynamic stabilization by a quadrupole field is most effective, while for more critical (from the viewpoint of linear theory) harmonics with $n \gg 1$, the effect of a toroidal geometry is unessential. If necessary, the stability of the annulus can be analyzed with the aid of the Routh function obtained, along the same lines as the stability of a straight filament.

2. The equations of motion for mechanical variables have the form

$$\rho_0'' + 2\nu^{-2}[2(2-K) + \gamma(K-1) - \cos 2\tau]\rho_0 = -2a\nu^{-2}\cos 2\tau, \tag{2.1}$$
$$\tau = \omega t,$$

$$\rho_k'' + {}^1/_4\,(ka)^2\nu^{-2}[2\,(2-K) - \cos\,2\tau]\,\rho_k = 0,\ \ \tau = \omega t, \tag{2.2}$$

$$x_0'' - 16\varkappa_{q0}\nu^{-2}x_0\,\cos 2\tau = 0, \qquad 2\tau = \omega t, \tag{2.3}$$

$$z_0'' + 16\varkappa_{q0}\nu^{-2}z_0\,\cos 2\tau = 0, \qquad 2\tau = \omega t, \tag{2.4}$$

$$x_k'' - 2\nu^{-2}\,[(ka)^2\,\Lambda_k + 8\varkappa_{q0}\cos 2\tau + (ka)^2(\Lambda_k + 2 - K)\cos\,4\tau]x_k = 0, \quad 2\tau = \omega t, \tag{2.5}$$

$$z_k'' - 2\nu^{-2}[(ka)^2\Lambda_k - 8\varkappa_{q0}\cos\,2\tau + (ka)^2\,(\Lambda_k + 2 - K)\cos 4\tau]z_k = 0, \quad 2\tau = \omega t. \tag{2.6}$$

Here,

$$\nu = \omega\,/\,\Omega, \quad \Omega^2 = 2p_{2c}\,/\,(\sigma a^2), \quad \Lambda_k = 2\ln\,(2/ka) - 2C + K - 3,$$

while the prime denotes differentiation with respect to τ. Equations (2.1)–(2.6) can be reduced to the standard form of Hill's equation [3]

$$u'' + (\theta_0 + 2\theta_1\,\cos 2\tau + 2\theta_2\,\cos\,4\tau)\,u = 0; \tag{2.7}$$

Eq. (2.1) differs from the other equations only by the presence of a right-hand side.

Let us examine first serpentine-type oscillations (2.3)–(2.6). The coefficients θ_0, θ_1, and θ_2 can be written at once for any wave number k:

$$\theta_0 = -2\,(ka)^2\nu^{-2}\Lambda_k, \quad \theta_1 = \pm\,8\varkappa_{q0}\nu^{-2}, \quad \theta_2 = -(ka)^2\nu^{-2}\,(\Lambda_k + 2 - K). \tag{2.8}$$

For oscillations with respect to z and θ_1, one should take the plus sign, and for oscillations with respect to x, the minus sign. In the case of a displacement of the filament as a whole, k = 0, $\theta_0 = \theta_2 = 0$, and in order to obtain stability [3], the inequality

$$|\theta_1| < 0.9. \tag{2.9}$$

must be satisfied.

It is true that there exist stability regions also for $|\theta_1| \gg 1$; however, they do not tolerate parameter variations within narrow limits, and are therefore of little practical interest. For disturbances at finite wavelengths (specifically serpentine disturbances), we have $\theta_0 < 0$, and provided that inequalities (2.9) and

$$|\theta_2| < 1 \tag{2.10}$$

are satisifed, the stability condition can be written with satisfactory accuracy [3] in the form

$$\theta_0 + {}^1/_2\,\theta_1^2\,(1 + {}^1/\ \theta_2) > 0. \tag{2.11}$$

By solving this inequality in combination with (1.20) and (2.10), and taking the concrete form of θ_0 and θ_2 into account, it can be shown that $|\theta_2| \le 1/2$ even for $ka < 1/4$ and that, therefore, the term containing θ_2 in (2.11) can be neglected. Then instead of (2.11) we have

$$\theta_0 + {}^1/_2\,\theta_1^2 > 0, \tag{2.12}$$

which together with (2.9) yields

$$\sqrt{2|\theta_0|} < |\theta_1| < 0.9. \tag{2.13}$$

By substituting θ_0 and θ_1 from (2.8) into (2.13), we get

$$\nu > \nu_0 \equiv 2.2ka\,\sqrt{\Lambda_k}, \tag{2.14}$$

$$0.9\nu_0\nu < 8\varkappa_{q0} < 0.9\nu^2. \tag{2.15}$$

Proceeding from (2.14) and (2.15), it is possible to determine the current frequency ω in the filament and the gradient G_{q0} of the quadrupole field that are sufficient for stabilizing serpentine-type disturbances.

Let us turn now to the analysis of radial oscillations of the filament. As distinct from a system with direct current, in which case radial oscillations are described by equations with constant coefficients, in the case under consideration—just as for serpentine disturbances—the corresponding equations are of the (2.7) type.

It follows from (2.2) that for constrictions ($k \neq 0$),

$$\theta_0 = {}^1/_2 (ka)^2 \nu^{-2} (2 - K), \quad \theta_1 = -{}^1/_8 (ka)^2 \nu^{-2}, \quad \theta_2 = 0. \tag{2.16}$$

If condition (2.14) is satisfied, it can be readily shown that $|\theta_1| \ll 1$. In this case, for $K \geq 2$, where $\theta_0 \leq 0$, the stability criterion has the form (2.12). By substituting $\theta_0 = 4\theta_1(K-2)$ into (2.12), we obtain for K the inequalities

$$0 \leqslant K - 2 < {}^1/_8 \, |\theta_1| \ll {}^1/_8 .$$

For a filament with direct current [2], the constrictions are stable for $K < 2$, which, by virtue of (1.19), is equivalent to the well-known condition $p_1 > p_0$.

Without attaching undue importance to a small increase in the maximum permissible value of K, let us examine the consequences of $K \leq 2$ for a filament with an alternating current. In this case, $\theta_0 \geq 0$ and, consequently, a parametric excitation of the constrictions is possible. Since for condition 2.14) $|\theta_1| \ll 1$, in order to obtain stability one must satisfy [3], as an example, the inequality $1 - |\theta_1| > \theta_0$, which with allowance for (2.16) yields

$$\nu^2 > {}^1/_8 (ka)^2 [1 + 4(2 - K)] .$$

The latter inequality is satisfied as soon as condition (2.14) is fulfilled. This means that if $K \leq 2$ (i.e., $p_1 \geq p_0$), a filament stable with respect to serpentine disturbances is also stable with respect to constrictions.

Finally, let us examine the oscillations of the filament radius ($k = 0$). Equation (2.1) has the coefficients

$$\theta_0 = 2\nu^{-2} [2(2 - K) + \gamma (K - 1)], \quad \theta_1 = -\nu^{-2}, \quad \theta_2 = 0, \tag{2.17}$$

and a right-hand side proportional to $\cos 2\tau$. This means that the filament undergoes forced oscillations at a frequency of 2ω. However, an external force of this type is not a resonant force [4], and therfore stability is defined solely by the coefficients (2.17) of a homogeneous equation.

By writing θ_0 in the form $\theta_0 = 2\nu^{-2}[3 - K + (\gamma - 1)(K - 1)]$ we verify that $\theta_0 > 0$ holds for any value of K within the range (1.20), since $\gamma > 1$. As in the case of constrictions, a parametric build-up of oscillations can also occur in this case. The width of parametric excitation regions, as we know [3], increases with increasing ratio $\theta_0/|\theta_1|$. In the case under consideration, $\theta_0/|\theta_1| = 2[2(2-K) + \gamma(K-1)]$ and, if $2 \geq K > 1$ and $\gamma \leq 2$, then

$$2\gamma \leqslant \theta_0 / |\theta_1| \leqslant 4 .$$

Hence, the instability regions will be widest for $\theta_0 = 2\gamma|\theta_1|$, where $K = 2$.

For this case, setting $\gamma = 5/3$, we define the boundaries of the first three parametric excitation regions and their relative half-widths:

$$S \equiv \frac{\nu_{\max} - \nu_{\min}}{\nu_{\max} + \nu_{\min}}$$

$$2.08 > \nu > 1.53, \quad S = 0.15 \quad (1 \text{ region}),$$
$$0.91 > \nu > 0.82, \quad S = 0.05 \quad (2 \text{ region}),$$
$$0.605 > \nu > 0.580, \quad S = 0.02 \quad (3 \text{ region}).$$

Most hazardous is the first, widest instability region. However, by selecting ν slightly above its critical value as defined by criterion (2.14), we get $\nu < 1$, so that the operating point does not come to lie in the first region. The remaining regions are relatively narrow, so that the probability of higher-order parametric resonance is small. If need occurs, an undesirable resonance can be "tuned out" by slightly changing a system parameter

Thus, the filament may be considered stable with respect to longwave serpentine- and constriction-type disturbances if the conditions (2.14) and (2.15) are satisfied and $K \leq 2$ ($p_1 \geq p_0$). In the figure (right branch), stability regions of the filament are plotted for $K = 2$, which for a given value of ka lie between a parabola and the corresponding straight line. It is obvious that the values of ν and $\varkappa_{q0}$ required for stabilization increase as the wavelength of the disturbance decreases. For $K < 2$, the stability regions contract slightly, owing to an increase in the slope angle of the straight lines.

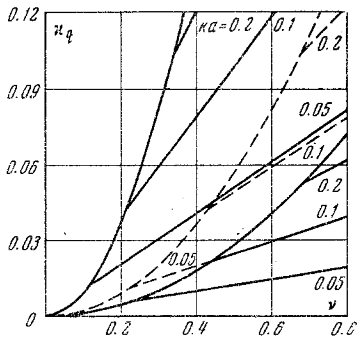

Fig. 1

3. Let the quadrupole field now vary at the same frequency as the filament current but with a phase shift of a quarter-period, i.e., if the current has the previous form (1.12), then

$$B_q = B_{qs} \sin \omega t, \quad G_q = G_{qs} \sin\omega t$$

and hence

$$\Phi_q = \Phi_{qs} \sin\omega t.$$

The symmetry of the x and z oscillations is not disrupted at the closest phase drifts.

The flux Φ should be taken in a more general form than in Eq. (1.13).

$$\Phi_0 = \Phi_{00} + \Phi_{0c} \cos\omega t + \Phi_{0s} \sin\omega t.$$

Then, instead of (1.14) we obtain

$$\Phi_{00} + \Phi_{0c} \cos\omega t + \Phi_{0s} \sin \omega t + \Phi_{qs} \sin \omega t - \Phi_2 = -2\pi blc^{-1} I_{2c} \cos\omega t$$

From here, we have

$$\Phi_{00} - \Phi_2 = 0, \quad \Phi_{0s} + \Phi_{qs} = 0, \quad \Phi_{0c} = -2\pi blc^{-1} I_{2c}.$$

Apparently, the flux Φ_{00} may be absent altogether ($\Phi_2 = 0$), while Φ_q must be now compensated for by the alternating flux $\Phi_{0s} \sin \omega t$. In Eqs. (1.17), (1.21), and (1.23), $G_{q0} \cos \omega t$ must be replaced by $(1/2) G_{qs} \sin 2\omega t$.

As in the case of a constant quadrupole field, the radial oscillations of a straight filament are described by Eqs. (2.1) and (2.2), while the equations for serpentine-type oscillations take the form

$$x_0'' - 2\varkappa_{qs} \nu^{-2} x_0 \sin 2\tau = 0, \tag{3.1}$$

$$z_0'' + 2\varkappa_{qs} \nu^{-2} z_0 \sin 2\tau = 0, \tag{3.2}$$

$$x_k'' - {}^1/_2 \nu^{-2} [(ka)^2 \Lambda_k + 4\varkappa_{qs} \sin 2\tau + (ka)^2 (\Lambda_k + 2 - K) \cos 2\tau] x_k = 0, \tag{3.3}$$

$$z_k'' - {}^1/_2 \nu^{-2} [(ka)^2 \Lambda_k - 4\varkappa_{qs} \sin 2\tau + (ka)^2 (\Lambda_k + 2 - K) \cos 2\tau] z_k = 0, \tag{3.4}$$

where

$$\tau = \omega t, \quad \varkappa_{qs} = aG_{qs} / B_{2c}.$$

By an appropriate shift with respect to τ, Eqs. (3.1)–(3.4) reduce to the standard form (2.7). Their stability, therefore, can be analyzed along the same lines as in section 2 for a constant quadrupole field. Omitting the particulars, we write the filament stabilization conditions as follows:

$$K \leqslant 2, \ \nu > \nu_s \equiv 1.1ka \sqrt{\Lambda_k}, \quad 0.9\nu_s \nu < \varkappa_{qs} < 0.9\nu^2. \tag{3.5}$$

With respect to oscillations of the filament radius, conditions (3.5) hold with the same reservations as in section 2. The figure (left branch) shows the stability regions obtained on the basis of inequalities (3.5) for $K = 2$.

4. For comparison, we give the results of [2], which can be readily written for a straight filament: if

$$K < 2, \quad \nu > \nu_c \equiv 2{,}2ka\sqrt{\Lambda_L}, \quad 0.9\nu_c\nu < 4\varkappa_{qc} < 0.9\nu^2$$
$$(\varkappa_{qc} = aG_{qc}/B_2,\ B_2 = \text{const},\ G_q = G_{qc}\cos\omega t)\,, \tag{4.1}$$

then the filament is stable with respect to longwave serpentine- and constriction-type disturbances. In the figure (central branch, dashed curves), the stability regions of the filament are plotted on the basis of inequalities (4.1) for the limiting case K = 2.

From (2.14), (2.15), (3.5), and (4.1), one can obtain the relations

$$G_{qc} = 2G_{q0} = G_{qs}, \quad \omega_c = \omega_0 = 2\omega_s$$

for the lower bounds of the operating frequencies and quadrupolefield gradients. Here, the subscripts c and s refer to quantities associated with direct [2] and alternating (section 3) currents in an alternating quadrupole field, and the subscript 0 refers to the direct current in a constant quadrupole field (section 2). It can be seen that in comparison with the stabilizating method [2], in the case of an alternating current in the filament, one can halve the gradient of the stabilizing field when this field is constant, or halve the operating frequency when both the field and current are alternating. The first two versions, where either the current or the quadrupole field is constant, are, however, more attractive from the practical point of view.

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